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Wave functionals of free real and complex scalar fields on a $1+1$ dimensional lattice

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We obtain wave functionals of free real and complex scalar fields on a $1+1$ dimensional lattice by explicitly calculating the path integral for transition from one field configuration to another. The obtained expressions are useful for cross-checking quality of approximations schemes used to study self-interacting fields on the lattice.

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In cases when the perturbative methods are not applicable a powerful method to study properties of a system under consideration is to put it on the lattice and study its properties numerically. Examples include a real scalar field with double well potential,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \lambda(\varphi^2 - v^2)^2\,, \tag{1}$$

and a system of two scalar fields with Mexican hat potential,

$$\mathcal{L} = \frac{1}{2}\textstyle\sum_{i=1}^2\partial_\mu\varphi_i\,\partial^\mu\varphi_i - \lambda\big(\sum_{i=1}^2\varphi_i^2 - v^2\big)^2\,, \tag{2}$$

in the limit of very small but nonzero v. To cross check the quality of the numerical approximations one can use the same numerical method to analyze models with known analytical solutions. For (1) the cross check can be done using the lattice version of a free massive scalar field. On a $1+1$ dimensional Euclidean lattice of a spatial size L its Lagrangian reads

$$\mathcal{L}_\alpha = \frac{1}{2}\dot\varphi_\alpha^2 + \frac{1}{2}\nabla\varphi_\alpha^2 + \frac{m^2}{2}\varphi_\alpha^2\,, \tag{3}$$

where $\nabla\varphi_\alpha = (\varphi_\alpha - \varphi_{\alpha-1})/\Delta l$, $\Delta l = L/K$, K is the number of sites,

$$\varphi_\alpha(t) = \int\limits_{x_{\alpha-1}}^{x_\alpha}\frac{dx}{\Delta l}\,\varphi(t,x)\,, \tag{4}$$

and $x_\alpha = \alpha\Delta l$. As can be inferred from (3) we deal with a system of K coupled oscillators $\varphi_1(t)\ldots\varphi_K(t)$. The finiteness of the space introduces a periodicity condition $\varphi(t,x) = \varphi(t,x+L)$ and therefore $\varphi_K = \varphi_0$. The Lagrange functional is given by $\Lambda = \Delta l\sum_\alpha\mathcal{L}_\alpha$. To diagonalize it we introduce Fourier-transforms φ_k of φ_α according to

$$\varphi_\alpha(t) = K^{-\frac{1}{2}}\sum_{k=0}^{K-1}\exp(i\,p_k\,x_\alpha)\varphi_k(t)\,, \tag{5}$$

where $p_k = 2\pi k/L$. The requirement that $\varphi_\alpha(t)$ be real-valued implies that $\varphi_{K-k}(t) = \varphi_k^*(t)$ so that the number of independent degrees of freedom is again equal to K. The Jacobian of this transformation is unity. In the momentum representation the Lagrange function is given by a sum of Lagrange functions of independent oscillators,

$$\Lambda = \frac{\Delta l}{2}\sum_{k=0}^{K-1}\left(\dot\varphi_k^*\dot\varphi_k + \Omega_k^2\varphi_k^*\varphi_k\right)\,, \tag{6}$$

where $\Omega_k^2 = \Omega_{K-k}^2 = (2/\Delta l\sin(p_k\Delta l/2))^2 + m^2$ is the lattice version of the relativistic energy-momentum relation.

One of the ways to extract energy spectrum and wave functionals of (6) is to use spectral representation of the transition amplitude from one field configuration, $\varphi_{0,in}\ldots\varphi_{K-1,in}$, to another one, $\varphi_{0,fin}\ldots\varphi_{K-1,fin}$. The transition amplitude can be also computed using the path integral. Representing the paths as a sum of the classical trajectory and fluctuations we can write the latter in the form

$$\mathcal{A}_E = \exp(-S_{cl})\int\mathscr{D}\varphi\exp(-S_{fl})\,. \tag{7}$$

Solving the (Euclidean version of the) Lagrange equations of motion we find

$$\begin{aligned}\varphi_k(t) = [&\varphi_{k,fin}\sinh\Omega_k(t-t_{in})\\ &+\varphi_{k,in}\sinh\Omega_k(t-t_{fin})]/\sinh\Omega_k T\,,\end{aligned} \tag{8}$$

where $T\equiv t_{fin}-t_{in}$. The classical action then reads

$$\begin{aligned}S_{cl} = \Delta l\sum_{k=0}^{K-1}&\frac{\Omega_k}{2\sinh\Omega_k T}\left[\left(\varphi_{k,fin}\varphi_{k,fin}^* + \varphi_{k,in}\varphi_{k,in}^*\right)\right.\\ &\left.\times\cosh\Omega_k T - \varphi_{k,fin}^*\varphi_{k,in} - \varphi_{k,in}^*\varphi_{k,fin}\right]\,,\end{aligned} \tag{9}$$

The Lagrange function of the fluctuations also has the form (6) but with the boundary conditions $\varphi_k(t_{in}) = \varphi_k(t_{fin}) = 0$. The boundary conditions take into account that (by definition) the fluctuations vanish at $t = t_{in}$ and $t = t_{fin}$. To calculate action of the fluctuations it is convenient to split the time interval from t_{in} to t_{fin} into $N+1$ small intervals $\Delta t = T/(N+1)$ and Fourier-transform $\varphi_k(t)$ with respect to time. The boundary conditions for the fluctuations imply that the cosine term of the Fourier-transformation vanishes and we are left with

$$\varphi_k(t_n) = \sqrt{\frac{2}{N+1}}\sum_{m=1}^N\sin(\omega_m t_n)\varphi_{m,k}\,, \tag{10}$$

where $\omega_m = (\pi/T)\,m$. Jacobian of this transformation is again unity. Using (10) we can write the action of the fluctuations in a compact form

$$S_{fl} = \Delta t\Delta l\sum_{k=0}^{K-1}\sum_{m=1}^N\epsilon_{m,k}^2\,\varphi_{m,k}^*\varphi_{m,k}\,, \tag{11}$$

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where

$$\epsilon^2{}_{m,k} = \epsilon^2_{m,K-k} = \left(\frac{2}{\Delta t}\sin\frac{\omega_m \Delta t}{2}\right)^2 + \Omega_k^2\,. \tag{12}$$

Finally, the measure $\mathscr{D}\varphi$ in the path integral (7) is given by

$$\mathscr{D}\varphi = \left(\frac{\Delta l}{2\pi\Delta t}\right)^{\frac{K(N+1)}{2}} \prod_{k=0}^{K-1}\prod_{m=1}^{N} d\,\boldsymbol{\varphi}_{m,k}\,. \tag{13}$$

The symbol $\boldsymbol{\varphi}$ denotes independent degrees of freedom. As has been mentioned above, due to the periodicity condition components of φ_{K-k} and φ_k are related by $\varphi_{m,K-k} = \varphi^*_{m,k}$. This implies in particular that $\varphi_{m,0}$ is real-valued. Taking in addition into account the periodicity of the eigenvalues (12) it is convenient to choose the independent degrees of freedom in such a way that, for instance, $\boldsymbol{\varphi}_{m,k} = \mathrm{Re}\,\varphi_{m,k}$ and $\boldsymbol{\varphi}_{m,K-k} = \mathrm{Im}\,\varphi_{m,k}$, where $k < K/2$. This definition implies also that $\boldsymbol{\varphi}_{m,0} = \varphi_{m,0}$. Rewritten in terms of $\boldsymbol{\varphi}_k$, the action on the classical trajectory (9) takes the form

$$S_{cl} = \Delta l \sum_{k=0}^{K-1} \frac{\Omega_k}{2\sinh\Omega_k T}\left[\left(\boldsymbol{\varphi}^2_{k,fin} + \boldsymbol{\varphi}^2_{k,in}\right)\cosh\Omega_k T \right. \\ \left. -2\boldsymbol{\varphi}_{k,fin}\boldsymbol{\varphi}_{k,in}\right]\,. \tag{14}$$

For the action of the fluctuations we find

$$S_{fl} = \Delta t \Delta l \sum_{k=0}^{K-1}\sum_{m=1}^{N} \epsilon^2_{m,k}\,\boldsymbol{\varphi}_{m,k}\boldsymbol{\varphi}_{m,k}\,. \tag{15}$$

Since all $\epsilon^2_{m,k}$ are positive the surfaces of constant action $S_{fl} = \mathrm{const.}$ are ellipses in the $N\times K$ dimensional space. The result of the Gaussian integration is proportional to the product of $\epsilon^{-1}_{m,k}$. The product over m is given by

$$\int \mathscr{D}\varphi \exp(-S_{fl}) = \prod_{k=0}^{K-1}\left(\frac{1}{2\pi}\frac{\Delta l}{\Delta t}\frac{\sinh\Delta t\boldsymbol{\Omega}_k}{\sinh T\boldsymbol{\Omega}_k}\right)^{\frac{1}{2}}\,, \tag{16}$$

where $\boldsymbol{\Omega}_k \equiv \frac{2}{\Delta t}\mathrm{arcsinh}\frac{\Delta t \Omega_k}{2} \to \Omega_k$ as $\Delta t \to 0$. Combining (16) and (14) and taking the limit $\Delta t \to 0$ we finally find for the transition amplitude

$$\mathcal{A}_E = \prod_{k=0}^{K-1}\left(\frac{\Omega_k \Delta l}{2\pi\sinh\Omega_k T}\right)^{\frac{1}{2}} \exp\Bigg(-\frac{\Omega_k\Delta l}{2}\coth\Omega_k T \\ \times\left[\boldsymbol{\varphi}^2_{k,in} + \boldsymbol{\varphi}^2_{k,fin}\right]\Bigg)\exp\left(\frac{\Omega_k \Delta l}{\sinh\Omega_k T}\boldsymbol{\varphi}_{k,in}\boldsymbol{\varphi}_{k,fin}\right)\,. \tag{17}$$

The first factor on the right-hand side of (17) is the fluctuation determinant and the remaining two factors corresponds to the classical trajectory.

Making use of a summation formula for Hermite polynomials H_n [1] we can obtain the spectral representation of (17)

$$\mathcal{A}_E = \sum \langle\boldsymbol{\varphi}^{fin}|n_k\rangle\langle n_k|\boldsymbol{\varphi}^{in}\rangle \exp\left(-TE_{n_k}\right)\,. \tag{18}$$

The wave functionals are given by

$$\langle\boldsymbol{\varphi}|n_k\rangle = \prod_{k=0}^{K-1}\left(\frac{\Omega_k\Delta l}{\pi}\right)^{\frac{1}{4}}\exp\left(-\frac{\Omega_k\Delta l}{2}\boldsymbol{\varphi}_k^2\right) \\ \times\left(\frac{1}{2^{n_k}n_k!}\right)^{\frac{1}{2}} H_{n_k}\left([\Omega_k\Delta l\,]^{\frac{1}{2}}\,\boldsymbol{\varphi}_k\right), \tag{19}$$

and the energy levels by

$$E_{n_k} = \sum_{k=0}^{K-1}\Omega_k\left(n_k + \frac{1}{2}\right)\,. \tag{20}$$

The vacuum corresponds to $n_k = 0$ for all k. In this case the second line in (19) is equal to unity. Performing inverse Fourier transformation we find for the wave functional of the vacuum state

$$\langle\varphi|0\rangle = \mathcal{N}\exp\Bigg(-\frac{1}{2}\sum_{\alpha,\beta=1}^{K}\varphi_\alpha\Delta^{\alpha\beta}\varphi_\beta\Bigg)\,, \tag{21}$$

where φ is the original scalar field on the lattice. The normalization factor $\mathcal{N}$ and the matrix Δ are given by

$$\mathcal{N} \equiv \prod_{k=0}^{K-1}\left(\frac{\Omega_k\Delta l}{\pi}\right)^{\frac{1}{4}}\,, \tag{22a}$$

$$\Delta^{\alpha\beta} \equiv K^{-1}\sum_{k=0}^{K-1}\Omega_k\Delta l\cos\left(p_k[x_\alpha - x_\beta]\right)\,. \tag{22b}$$

The fact that $\Delta^{\alpha\beta}$ depends on the difference of x_α and x_β reflects translational invariance of the vacuum. Using Fourier representation of the delta-function on the lattice and the definition of a function of an operator we can represent it in the form $\sqrt{-\nabla^2 + m^2}\,\delta(x_\alpha - x_\beta)$. It is identical to the form that can be obtained for the real scalar field using the Schrödinger representation [2].

For (2) the cross check can be done using the lattice version of a system of two free massive scalar fields with equal masses, which is equivalent to a free complex scalar field. On a $1+1$ dimensional Euclidean lattice its Lagrangian reads

$$\mathcal{L}_\alpha = \frac{1}{2}\sum_{i=1}^{2}\dot\varphi^2_{i,\alpha} + \frac{1}{2}\sum_{i=1}^{2}\nabla\varphi^2_{i,\alpha} + \frac{m^2}{2}\sum_{i=1}^{2}\varphi^2_{i,\alpha}\,. \tag{23}$$

Because the two field degrees of freedom are independent the resulting transition amplitude is given by a product of the two amplitudes (17). Introducing the radial and angular components, $\boldsymbol{r}$ and $\boldsymbol{\theta}$, and using the relation $\exp(\boldsymbol{r}\cos\boldsymbol{\theta}) = \sum_{m=-\infty}^{\infty} I_m(\boldsymbol{r})\exp(im\boldsymbol{\theta})$ [1] we can represent it in the form

$$\mathcal{A}_E = \prod_{k=0}^{K-1}\left(\frac{\Omega_k\Delta l}{2\pi\sinh\Omega_k T}\right)\exp\Bigg(-\frac{\Omega_k\Delta l}{2}\coth\Omega_k T\left[\boldsymbol{r}^2_{k,in}\right. \\ \left. + \boldsymbol{r}^2_{k,fin}\right]\Bigg)\times\sum_{m_k=-\infty}^{\infty} I_{m_k}\left(\frac{\Omega_k\Delta l}{\sinh\Omega_k T}\boldsymbol{r}_{k,in}\boldsymbol{r}_{k,fin}\right) \\ \times\exp\left(im_k[\boldsymbol{\theta}_{k,fin} - \boldsymbol{\theta}_{k,in}]\right)\,. \tag{24}$$

Making use of the Hille-Hardy formula [3] we can extract from (24) the energy spectrum of the system,

$$E = \sum_{k=0}^{K-1} \Omega_k \left(2n_k + m_k + 1\right) \, , \tag{25}$$

as well as the form of the corresponding wave functionals,

$$\begin{aligned} \langle \boldsymbol{r}, \boldsymbol{\theta} | \, n_k, m_k \rangle = & \prod_{k=0}^{K-1} \left(\frac{\Omega_k \Delta l}{\pi} \right)^{\frac{1}{2}} \exp\left(-\frac{\Omega_k \Delta l}{2} \boldsymbol{r}_k^2 \right) \\ & \times \left(\frac{n_k!}{\Gamma(n_k + m_k + 1)} \right)^{\frac{1}{2}} \left(\Omega_k \Delta l \, \boldsymbol{r}_k^2 \right)^{\frac{m_k}{2}} L_{n_k}^{(m_k)} \left(\Omega_k \Delta l \, \boldsymbol{r}_k^2 \right) \\ & \times \exp(i m_k \boldsymbol{\theta}_k) \, , \end{aligned} \tag{26}$$

where $L_n^{(m)}$ are the generalized Laguerre polynomials. The vacuum corresponds to $n_k, \, m_k = 0$ for all k. In this case the second and third lines in (26) are equal to unity and the wave functional is cylindrically symmetric in the Fourier space. Performing inverse Fourier transformation we find for the wave functional of the vacuum

$$\langle r, \theta | 0 \rangle = \mathcal{N}^2 \exp\left(-\frac{1}{2} \sum_{\alpha,\beta=1}^{K} r_\alpha \Delta^{\alpha\beta} r_\beta \cos(\theta_\alpha - \theta_\beta) \right), \tag{27}$$

where the radial and angular variables are related to the field components by $\varphi_1 = r \cos\theta$ and $\varphi_2 = r \sin\theta$. The fact that the wave function depends on the difference of the field phases θ indicates that the vacuum is cylindrically symmetric.

To summarize, we have calculated energy spectra and wave functionals of free real and free complex scalar fields on a 1+1 dimensional space using the path integral method. The obtained expressions are useful for cross-checking quality of approximations schemes that are used to study self-interacting fields on the lattice.

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